1. Starting with $X$, from the Durgin paper,

$$f_X(x) = \begin{cases} 
\frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0 \\
0, & \text{o.w.}
\end{cases}$$

Since $g(x) = y = x^2$, then $g^{-1}(y) = \pm \sqrt{y}$. The derivative of the inverse function w.r.t. $y$ is

$$\frac{\partial g^{-1}}{\partial y}(y) = \pm \frac{1}{2\sqrt{y}}$$

Plugging into the Jacobian formula,

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right|$$

$$= \begin{cases} 
\pm \frac{\sqrt{y}}{\sigma^2} e^{-\frac{y}{2\sigma^2}} \frac{1}{2\sqrt{y}}, & y \geq 0 \\
0, & \text{o.w.}
\end{cases}$$

$$= \begin{cases} 
\frac{1}{2\sigma^2} e^{-\frac{y}{2\sigma^2}}, & y \geq 0 \\
0, & \text{o.w.}
\end{cases}$$

Which is the same as an exponential pdf with parameter $\lambda = \frac{1}{2\sigma^2}$. Note that the condition $\pm \sqrt{y} > 0$ in line simplifies to $y > 0$, and eliminates the possibility of $-\sqrt{y}$. Thus we can use only $g^{-1}(y) = + \sqrt{y}$.

Now, let’s find the distribution of $Z = 10 \log_{10} Y$. Finding the inverse, $y = g^{-1}(z) = 10^{z/10}$. Note that this can also be written,

$$g^{-1}(z) = \exp \left( \frac{z \ln 10}{10} \right)$$

Taking the derivative,

$$\frac{\partial g^{-1}}{\partial z}(z) = \frac{\ln 10}{10} \exp \left( \frac{z \ln 10}{10} \right) = \frac{\ln 10}{10} 10^{z/10}$$

Plugging in this into the Jacobian formula,

$$f_Z(z) = f_Y(g^{-1}(z)) \left| \frac{\partial g^{-1}(z)}{\partial z} \right|$$

$$= \begin{cases} 
\frac{\ln 10}{20\sigma^2} 10^{z/10} e^{-\frac{1}{2\sigma^2} 10^{z/10}}, & \frac{\ln 10}{10} 10^{z/10} \geq 0 \\
0, & \text{o.w.}
\end{cases}$$

This is the log-Rayleigh distribution.
Now let’s use the same dB transform on $X$, that is, $W = 10 \log_{10} X$. We’ve already computed the inverse function and its derivative. Plugging into the Jacobian formula,

$$f_W(w) = f_X(g^{-1}(w)) \left| \frac{\partial g^{-1}(w)}{\partial w} \right|$$

(2)

$$= \begin{cases} \frac{10^{w/10}}{\sigma^2} e^{-\frac{1}{2\sigma^2} 10^{2w/10}} \ln 10 10^{w/10}, & 10^{w/10} \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

$$= \ln 10 10^{w/5} e^{-\frac{1}{2\sigma^2} 10^{w/5}}$$

Finally, $W = Z/2$. Thus $z = g^{-1}(w) = 2w$, and its derivative is $\frac{\partial}{\partial z} g^{-1}(w) = 2$. Thus applying the Jacobian method and (2),

$$f_W(w) = f_Z(g^{-1}(w)) \left| \frac{\partial g^{-1}(w)}{\partial w} \right|$$

(3)

$$= \ln 10 20^{2w/10} 10^{w/10} e^{-\frac{1}{2\sigma^2} 10^{2w/10}} 2$$

$$= \ln 10 10^{w/5} e^{-\frac{1}{2\sigma^2} 10^{w/5}}$$

which is the same as (2) above.

2. From [Durgin 2002],

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2 + V^2}{2\sigma^2}} I_0 \left( \frac{V x}{\sigma^2} \right), & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

Letting $Y = 10 \log_{10} X$, we have the same inverse function as derived above. Plugging in this into the Jacobian formula,

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right|$$

(3)

$$= \frac{10^{y/10}}{\sigma^2} e^{-\frac{10^{y/5} + V^2}{2\sigma^2}} I_0 \left( \frac{V_1 10^{y/10}}{\sigma^2} \right) \ln 10 10^{y/10}$$

$$= \ln 10 10^{y/5} e^{-\frac{10^{y/5} + V^2}{2\sigma^2}} I_0 \left( \frac{V_1 10^{y/10}}{\sigma^2} \right)$$

This is the log-Rician distribution. Plots are shown in Figure 1.

3. Scatterers are uniformly distributed on the perimeter of a square, so consider one such scatterer as shown in Figure 2.

First, let’s find the CDF. This is the probability $\theta$, the AOA of the path from the scatterer at the receiver, is between 0 and some other value, let’s call it $t$:

$$F_\theta(t) = P \left[ 0 \leq \theta < t \right]$$

The perimeter is uniformly likely, so the probability that $0 \leq \theta < t$ is the probability the scatterer is on the part of the perimeter with length labeled $x$. The probability is
Figure 1: Six different log-Rician probability density functions. Note that the mean shifts up when the total power increases, and the shape around that mean is determined by the $K$ factor, $K = \frac{V_1^2}{\sigma^2}$.

Figure 2: A scatterer is uniformly randomly distributed along the perimeter of a square.
the length $x$ divided by $4L$, the total length of the perimeter. Also, using trigonometry on Figure 2 and assuming $0 < t < \pi/4$, we know that $\tan t = \frac{x}{L/2}$, or equivalently $x = (L/2) \tan t$. Thus for $0 < t < \pi/4$,

$$F_\theta(t) = \frac{(L/2) \tan t}{4L} = \frac{1}{8} \tan t$$

The pdf is the derivative w.r.t. $t$,

$$f_\theta(t) = \frac{1}{8} \sec^2 t$$

Continuing for $\pi/4 < t < \pi/2$, we could use trig to find the total perimeter inside of the sector of $0 \leq \theta < t$ is

$$L - \frac{L}{2} \tan \left(\frac{\pi}{2} - t\right)$$

Thus for $\pi/4 < t < \pi/2$,

$$F_\theta(t) = \frac{L - (L/2) \tan \left(\frac{\pi}{2} - t\right)}{4L} = \frac{1}{4} - \frac{1}{8} \tan \left(\frac{\pi}{2} - t\right)$$

The pdf is the derivative w.r.t. $t$,

$$f_\theta(t) = \frac{1}{8} \sec^2 \left(\frac{\pi}{2} - t\right)$$

Beyond, for $t > \pi/2$, the pdf will repeat (it is the same geometry turned 90 degrees). Thus for all $t$ I can say

$$f_\theta(t) = \begin{cases} 
\frac{1}{8} \sec^2 \left(t \mod \frac{\pi}{2}\right), & t \mod \frac{\pi}{2} < \pi/4 \\
\frac{1}{8} \sec^2 \left(\frac{\pi}{2} - t \mod \frac{\pi}{2}\right), & t \mod \frac{\pi}{2} > \pi/4 
\end{cases}$$

Figure 3: The AOA distribution is highest towards the corners, in this case, at $-3\pi/4$, $-\pi/4$, $\pi/4$, and $3\pi/4$ radians.

4. Your results may vary. I will post some examples later.