ECE 6960: Adv. Random Processes & Applications
Lecture Notes, Fall 2010

Lecture 12

Today: Random Processes: (1) Indep. Increments, (2) Autocovariance, (3) Stationarity

- Homework 5: short (2 questions), due Thursday, but I assigned it late, so if you want to turn it in after break (Monday or Tuesday) that is fine.

1 Random Processes

1.1 Continuous and Discrete-Time

Types of Random Processes: A random process (R.P.) can be either

1. **Discrete-time**: Samples are taken at particular time instants, for example, $t_n = nT$ where n is an integer and T is the sampling period. In this case, rather than referring to $X(t_n)$, we abbreviate it as X_n. (This matches exactly our previous notation.) In this case, we also call it a *random sequence*.

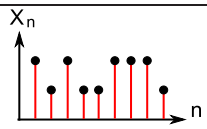
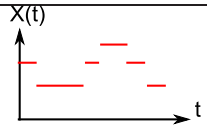
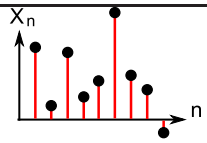
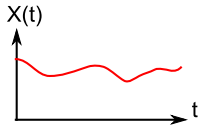
2. **Continuous-time**: Uncountably-infinite values exist, for example, for $t \in (0, \infty)$.

Types of Random Processes: A random process (R.P.) can be still be

1. **Discrete-valued**: The sample space S_X is countable. That is, each value in the R.P. is a discrete r.v. (For example, our R.P. can only take integer values, or we allow a finite number of decimal places.)

2. **Continuous-valued**: The sample space S_X is uncountably infinite.

Several examples of random processes are given in Section 9.1 and 9.2 of Leon-Garcia.

	Discrete-Time	**Continuous-Time**
Discrete-Valued		
Continuous-Valued		

1.2 Independent Increments

<u>Def'n:</u> *Independent Increments*
A cts. time r.p. $X(t)$ has the independent increments property if for two non-overlapping intervals (t_1, t_2) and (t_3, t_4) with $t_1 \leq t_2 \leq t_3 \leq t_4$, the changes $X(t_2) - X(t_1)$ and $X(t_4) - X(t_3)$ are independent.

The definition for discrete-time r.p.s is the same, but using integer times $n_1 \leq n_2 \leq n_3 \leq n_4$, the the changes $X_{n_2} - X_{n_1}$ and $X_{n_4} - X_{n_3}$ are independent.

A simple example of an independent increments r.p. X_n is one which is a sum of an i.i.d. random process Y_n. Let

$$X_n = Y_1 + \cdots + Y_n = \sum_{i=1}^{n} Y_i$$

Since

$$\begin{aligned} X_{n_2} - X_{n_1} &= Y_{n_1+1} + \cdots + Y_{n_2} \\ X_{n_4} - X_{n_3} &= Y_{n_3+1} + \cdots + Y_{n_4} \end{aligned}$$

don't depend on any of the same r.v.s Y_i, and since Y_i, Y_j are independent for $i \neq j$, thus the two intervals are independent.

We will show next lecture that the Poisson process has the independent increments property.

1.3 Expected Value and Correlation

<u>Def'n:</u> *Expected Value of a Random Process*
The expected value of continuous time random process $X(t)$ is the deterministic function

$$\mu_X(t) = E_{X(t)}\left[X(t)\right]$$

for the discrete-time random process X_n,

$$\mu_X[n] = E_{X_n}\left[X_n\right]$$

Example: What is the expected value of a Poisson process?
A Poisson process has arrival count pmf $p_X(x) = \frac{(\lambda t)^x}{x!}e^{-\lambda t}$ for some rate of arrivals λ, for all time $t > 0$. What is the mean number of arrivals, as a function of t?

$$\begin{aligned}
\mu_X(t) &= E_{X(t)}\left[X(t)\right] = \sum_{x=0}^{\infty} \frac{(\lambda t)^x}{x!} e^{-\lambda t} \\
&= e^{-\lambda t} \sum_{x=0}^{\infty} x \frac{(\lambda t)^x}{x!} \\
&= e^{-\lambda t} \sum_{x=1}^{\infty} \frac{(\lambda t)^x}{(x-1)!} \\
&= (\lambda t) e^{-\lambda t} \sum_{x=1}^{\infty} \frac{(\lambda t)^{x-1}}{(x-1)!} \\
&= (\lambda t) e^{-\lambda t} \sum_{y=0}^{\infty} \frac{(\lambda t)^{y}}{(y)!} \\
&= (\lambda t) e^{-\lambda t} e^{\lambda t} = \lambda t \qquad (1)
\end{aligned}$$

This is how we intuitively started deriving the Poisson process - we said that it is the process in which on average we have λ arrivals per unit time. Thus we'd certainly expect to see λt arrivals after a time duration t.

Example: What is the expected value of X_n, the number of successes in a Bernoulli process after n trials?
We know that X_n is Binomial, with mean np. This is the mean function, if we consider it to be a function of n: $\mu_X[n] = E_X\left[X_n\right] = np$.

1.4 Autocovariance and Autocorrelation

These next two definitions are the most critical concepts of the rest of the semester. Generally, for a time-varying signal, we often want to know two things:

- How to predict its future value. (It is not deterministic.) We will use the 'autocovariance' to determine this.
- How much 'power' is in the signal (and how much power within particular frequency bands). We will use the 'autocorrelation' to determine this.

<u>Def'n:</u> *Autocovariance*
The autocovariance function of the continuous time random process $X(t)$ is

$$C_X(t,\tau) = \text{Cov}\left(X(t), X(t+\tau)\right)$$

For a discrete-time random process, X_n, it is

$$C_X[m,k] = \text{Cov}\left(X_m, X_{m+k}\right)$$

Note that $C_X(t,0) = \text{Var}_X\left[X(t)\right]$, and $C_X[m,0] = \text{Var}_X\left[X_m\right]$.

<u>Def'n:</u> *Autocorrelation*
The autocorrelation function of the continuous time random process $X(t)$ is

$$R_X(t,\tau) = E_X\left[X(t)X(t+\tau)\right]$$

For a discrete-time random process, X_n, it is

$$R_X[m,k] = E_X\left[X_m X_{m+k}\right]$$

These two definitions are related by:

$$\begin{aligned} C_X(t,\tau) &= R_X(t,\tau) - \mu_X(t)\mu_X(t+\tau) \\ C_X[m,k] &= R_X[m,k] - \mu_X[m]\mu_X[m+k] \end{aligned}$$

Example: Poisson R.P. Autocorrelation and Autocovariance
What is $R_X(t,\tau)$ and $C_X(t,\tau)$ for a Poisson R.P. $X(t)$?

$$R_X(t,\tau) = E_X\left[X(t)X(t+\tau)\right]$$

Look at this very carefully! It is a product of two samples of the Poisson R.P. Are $X(t)$ and $X(t+\tau)$ independent?

- NO! They are not – $X(t)$ counts the arrivals between time 0 and time t. $X(t+\tau)$ counts the arrivals between time 0 and time $t+\tau$. The intervals $(0,t)$ and $(0,t+\tau)$ do overlap! So they are not the "non-overlapping intervals" required to apply the independent increments property.

See Figure 1. But, we can re-work the product above to include two terms which allow us to apply the independent increments property. WATCH THIS:

$$R_X(t,\tau) = E_X\left[X(t)[X(t+\tau) - X(t) + X(t)]\right]$$

Note that because of the independent increments property, $[X(t+\tau) - X(t)]$ is independent of $X(t)$.

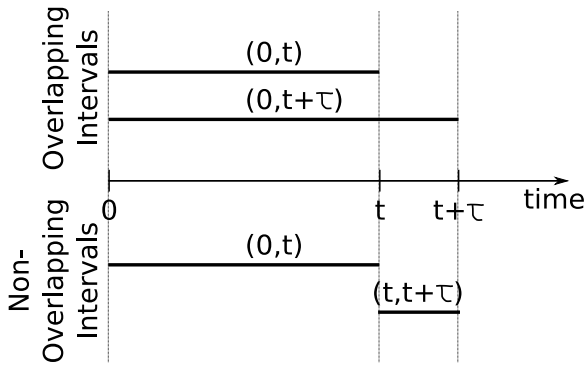


Figure 1: Comparison of non-overlapping and overlapping intervals. Converting problems which deal with overlapping intervals of time to a form which is in terms of non-overlapping intervals is a key method for analysis.

$$R_X(t,\tau) = E_X\left[X(t)[X(t+\tau)-X(t)]\right] + E_X\left[X(t)X(t)\right] \tag{2}$$

See what we did? The first expected value is now a product of two r.v.s which correspond to non-overlapping intervals. LEARN THIS TRICK! This one trick, converting products with non-independent increments to products of independent increments, will help you solve a lot of of the autocovariance problems you'll see.

$$\begin{aligned} &= E_X\left[X(t)\right] E_X\left[X(t+\tau)-X(t)\right] + E_X\left[X^2(t)\right] \\ &= \mu_X(t)\mu_X(\tau) + E_X\left[X^2(t)\right] \\ &= \mu_X(t)\mu_X(\tau) + \mathrm{Var}_X\left[X(t)\right] + \left[\mu_X(t)\right]^2 \end{aligned} \tag{3}$$

We will not derive it right now, but $\mathrm{Var}_X\left[X(t)\right] = \lambda t$. So

$$R_X(t,\tau) = \lambda t \lambda \tau + \lambda t + \lambda^2 t^2 = \lambda t\left[\lambda(t+\tau)+1\right]$$

Then

$$\begin{aligned} C_X(t,\tau) &= R_X(t,\tau) - \mu_X(t)\mu_X(t+\tau) \\ &= \lambda t\left[\lambda(t+\tau)+1\right] - \mu_X(t)\mu_X(t+\tau) \\ &= \lambda t\left[\lambda(t+\tau)+1\right] - \lambda t \lambda(t+\tau) \\ &= \lambda t \end{aligned}$$

The autocovariance at time t is the same as the variance of $X(t)$! We will revisit this later, but all random processes which have the independent increments property exhibit this trait.

Example: Example 9.10
We select a phase Θ to be uniform on $[0, 2\pi)$. Define:

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

What are the mean and covariance functions of $X(t)$?
Solution: Note that we need two facts for this derivation. First, for any integer k, real valued α,

$$E_\Theta\left[cos(\alpha + k\Theta)\right] = 0$$

Also, we'll need the identity $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$.

Because of the first fact,

$$\mu_X(t) = 0$$

From the 2nd fact,

$$\begin{aligned}
C_X(t,\tau) &= R_X(t,\tau) = E_X\left[A\cos(2\pi f_c t + \Theta) A\cos(2\pi f_c(t+\tau) + \Theta)\right] \\
&= \frac{A^2}{2} E_X\left[\cos(2\pi f_c \tau) + \cos(2\pi f_c(2t+\tau) + 2\Theta)\right] \\
&= \frac{A^2}{2} E_X\left[\cos(2\pi f_c \tau) + \cos(2\pi f_c(2t+\tau) + 2\Theta)\right] \\
&= \frac{A^2}{2} E_X\left[\cos(2\pi f_c \tau)\right] \\
&= \frac{A^2}{2}\cos(2\pi f_c \tau)
\end{aligned}$$

1.5 Wide Sense Stationary

<u>Def'n:</u> *Wide Sense Stationary (WSS)*
A R.P. $X(t)$ is wide-sense stationary if its mean function and covariance function (or correlation function) is **not** a function of t, *i.e.*,

$$E_X\left[X(t)\right] = \mu_X, \quad \text{and} \quad R_X(t,\tau) = R_X(s,\tau) \doteq R_X(\tau)$$

and

$$E_X\left[X_n\right] = \mu_X, \quad \text{and} \quad R_X[m,k] = R_X[n,k] \doteq R_X[k]$$

Intuitive Meaning: The mean does not change over time. The covariance and correlation of a signal with itself at a time delay does not change over time.

Note that R_X and μ_X not a function of t also means that C_X is not a function of t.

Example: WSS of past two examples
Are the Poisson process and/or Random phase sine wave process WSS? Solution:

- Poisson process: no.
- Random phase sine wave process: yes.

Processes that we'll see on a circuit, for example, we want to know that they have certain properties. WSS is one of them.

1.5.1 Properties of a WSS Signal

- $R_X(0) \geq 0$.
- $R_X(0)$ ($R_X[0]$) is the average power of $X(t)$ (X_n). Think of $X(t)$ as being a voltage or current signal, going through a 1 Ω resistor. $R_X(0)$ is the power dissipated in the resistor.
- $R_X(0) \geq |R_X(\tau)|$.
- $R_X(\tau) = R_X(-\tau)$.

For example, what is the power of the random phase sine wave process? Answer: $\frac{A^2}{2}$, where A was the amplitude of the sinusoid.

1.6 Strict-sense Stationary

<u>Def'n:</u> *Strict Sense Stationary, or "Stationary"*
A R.P. $X(t)$ is strict-sense stationary if the joint pdf of $X(t)$ at any k times, $t, t+\tau_1, \ldots, t+\tau_{k-1}$, are not a function of t.

Strict-sense stationary implies wide-sense stationary, but strict-sense stationary means much more than WSS. It means that all moments (not just the first and second) stay the same, when considering the same relative time delays $\tau_1, \ldots, \tau_{k-1}$. The "stationary" concept, like independence, is very difficult to prove. Thus you'd only see this property in certain random processes defined to have this property. In addition, we rarely deal with higher order moments (> 2) of random processes, so we don't typically need a r.p. to be stationary.