ECE 6960: Adv. Random Processes & Applications
Lecture Notes, Fall 2010

Lecture 15

Today: (0) Poisson, cont. (1) Gaussian Processes

1 Gaussian Processes

This is Leon-Garcia Section 9.5.

Def'n: *Gaussian random process*
A R.P. is a Gaussian random process if the samples $X(t_1), \ldots, X(t_k)$ are jointly Gaussian for all k and for all choices of $t_1, \ldots, t_k$.

A Gaussian R.P. is completely determined by its mean vector $\boldsymbol{\mu}$ and covariance matrix $C_{\mathbf{X}}$, and has pdf

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\exp -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T C_{\mathbf{X}}(\mathbf{x}-\boldsymbol{\mu})}{\sqrt{(2\pi)^k |C_{\mathbf{X}}|}}$$

Linear operations on Gaussian processes (*e.g.*, linear filters, sum, derivative, integral) also result in another Gaussian R.P.

White Gaussian noise is a good example of a Gaussian R.P. In particular, white Gaussian noise the case of i.i.d. Gaussian r.v.s with zero mean and constant variance.

2 Brownian Motion

Brownian motion is a continuous time 'random walk' process. It can be seen as

- White Gaussian noise input into an integrator.
- A limit of a Markovian random walk as the time interval and the step size both approach zero.

For the second perspective, Leon-Garcia considers the Markov symmetric random walk process $X_\delta(t)$. This process $X_\delta(t)$ is piecewise constant, and adds either $\pm h$ every δ seconds. In other words, after each time unit δ, we take a "step" size h in either the positive or negative direction, with equal probabilities. This is shown in Figure 1, where the transition probabilities are all 1/2, and δ.

If the random process starts with $X_\delta(0) = 0$, what are the expected value and variance of $X_\delta(t)$?

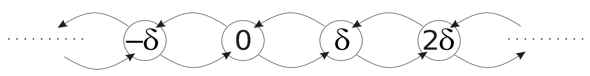

Figure 1: Markov random walk process.

Solution: The value of $X_\delta(t)$ is a sum of all of the steps taken between time 0 and time t. Denoting these (independent) transitions as $Y_i = X_\delta(i\delta) - X_\delta((i-1)\delta)$,

$$E\left[X_\delta(t)\right] = E\left[\sum_{i=1}^{t/\delta} Y_i\right] = 0$$

and the variance is

$$\text{Var}\left[X_\delta(t)\right] = \text{Var}\left[\sum_{i=1}^{t/\delta} Y_i\right] = \sum_{i=1}^{t/\delta} \text{Var}\left[Y_i\right] = \frac{t}{\delta}h^2.$$

Now, we choose to let h and δ go to zero as follows. Let $h = \sqrt{\alpha\delta}$ for some positive constant α and then let $\delta \to 0$. Doing so,

$$\lim_{\delta\to 0} \text{Var}\left[X_\delta(t)\right] = \lim_{\delta\to 0} \frac{t}{\delta}\alpha\delta = \alpha t.$$

Moreover, with the central limit theorem, it is intuitive that the sum of (infinitely) many independent contributions $\pm h$ will result in a Gaussian distribution for $X_\delta(t)$.

Def'n: *Brownian Motion Process*
A random process $\{X(t), t \geq 0\}$ is a Brownian motion process if

1. $X = 0$;
2. $\{X(t), t \geq 0\}$ has stationary and independent increments;
3. For every $t > 0$, $X(t)$ is Gaussian with zero-mean and variance αt.

Given a variance per unit time, α, we can form a multivariate pdf for any combinations of times $t_1, \ldots, t_n$ of the random process. Assuming we've ordered them from smallest to largest, the increments,

$$X(t_1) - 0, X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent. So we can take the product of n pdfs of these independent increments to describe the joint Brownian motion pdf at all n times:

$$f_{\mathbf{Y}}(y_1, \ldots, y_n) = f_{t_1}(y_1) f_{t_2-t_1}(y_2 - y_1) \cdots f_{t_n - t_{n-1}}(y_n - y_{n-1})$$

where $f_t(\cdot)$ indicates a Gaussian pdf with zero-mean and variance $\sigma^2 t$.

2.1 Financial Brownian Motion

- Brownian motion with drift process
- Geometric Brownian motion process

First, consider that we could have Brownian motion with drift. Instead of zero-mean Gaussian $X(t)$, we'd have a mean of μt. We could just define the new Brownian motion with drift process as $\{Y(t), t \geq 0\}$ by

$$Y(t) = X(t) + \mu t,$$

where $X(t)$ is a Brownian motion process with variance parameter $\alpha = \sigma^2$. We are just adding a deterministic function μt to a Brownian motion process.

Next, consider a geometric Brownian motion process $X(t)$ defined as

$$W(t) = e^{Y(t)}$$

where $Y(t)$ is a Brownian motion process with variance parameter α and drift parameter μ. Here, the changes in the Brownian motion with drift process multiply the process $W(t)$. The multipliers on non-overlapping segments are independent. In other words,

$$\log W(t) = Y(t)$$

the decibel (dB) changes in $\log W(t)$ are additive and independent.

This is a good model when you believe that the percentage change is constant. For example, in stock prices, the relative increases

$$Z_n = Y_n / Y_{n-1}$$

may be independent and stationary, but due to inflation, the absolute increases' mean and variance may differ over time.

Example: Mean Function of Geometric Brownian motion process
Given $W(u)$ for u from 0 to some time s, what is the expected value of $W(t)$?
Solution:

$$\begin{aligned} E\left[W(t)|W(u), 0 \leq u \leq s\right] &= E\left[e^{Y(t)}|W(s)\right] = E\left[e^{Y(t)-Y(s)+Y(s)}|W(s)\right] \\ &= E\left[e^{Y(t)-Y(s)}W(s)|W(s)\right] = W(s)E\left[e^{Y(t)-Y(s)}\right] \end{aligned}$$

Note now that the expected value function is a moment generating function. Because the difference in $Y(t)$ and $Y(s)$ is also Gaussian, we can write the MGF with the mean and variance of $Y(t) - Y(s)$. These are $\mu(t-s)$ and $\sigma^2(t-s)$ respectively. Thus,

$$E\left[W(t)|W(s)\right] = W(s) \exp\left[(t-s)\left(\mu + \frac{\sigma^2}{2}\right)\right]$$

It is surprising that even with $\mu = 0$, if $\sigma^2 > 0$, then the mean price of $W(t)$ is increasing with time. The multiplicative effect means that the spread in the positive direction is wider than the spread in the negative direction. Intuitively, you can't make the price lower than 0 (a reduction in value by 100%), but you can make the price rise more than 100%.

An option is something you purchase now, which gives you the option to either buy a stock at a time in the future for a set price, or not to buy the stock.

Let's call the stock price $X(t)$. Let's say the time now is s and the future time is t. You buy the option at time s at cost c, which gives you the option of buying the stock at price K at time t. But the price of the stock at time t, that is, $X(t)$, may be greater than or less than K. If $X(t) - K > 0$ then you will use your option to buy the stock and make a profit of $X(t) - K$. If $X(t) - K \leq 0$ then you will NOT use your option to buy the stock and end up with \$0. In short, you make $(X(t) - K)^+$ at time t, where $(x)^+$ is the notation for x if $x > 0$ – or equal to zero if $x \leq 0$. *If you're selling an option for a stock, how do you set the price c given the delay $t - s$ and the fixed price K?*

The Black-Scholes-Merton option pricing formula was derived in 1973 to provide a formula to price an option. Myron Scholes and Robert C. Merton, who expanded the model, won the 1997 Nobel prize in Economics for this work.

While the pricing of options is very complicated, as you can see from the handout, the basic idea is that the cost of buying an option should equal the expected value of the gain.

$$c = E_{X(t)}\left[e^{-\alpha t}(X(t) - K)^+\right] \tag{1}$$

The gain at time t is the $(X(t) - K)^+$ mentioned above, however, we convert all prices to "time 0" prices with the $e^{-\alpha t}$ to account for inflation at rate α.

To know the expected value at a future time, you must have a probability model for the value at a future time t. The Black-Scholes-Merton formula assumes that $X(t)$ is a Geometric Brownian motion R.P. With that assumption, one may find a solution to (1). The solution is given in the handout.

However, there are lots of problems with this plain model. One is that stock prices are said to be more volatile than Gaussian – the tails are heavier. That is, extreme drops and rises are more likely than the Gaussian pdf would indicate.